Spinor vortices
in
non-relativistic Chern-Simons theory

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Abstract.

The non-relativistic 'Dirac' equation of Lévy-Leblond is used to describe a spin 1/2 *particle interacting with a Chern-Simons gauge field. Static, purely magnetic, self-dual spinor vortices are constructed. The solution can be 'exported' to a uniform magnetic background field.*

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The non-relativistic Chern-Simons model of Jackiw and Pi [1] considers a massive scalar field, Ψ, described by the planar, gauged, non-linear Schrödinger equation,

$$iD_t\Psi = \left[-\frac{\vec{D}^2}{2m} - \Lambda\,\Psi^*\Psi\right]\Psi, \tag{1}$$

while the electromagnetic field and the current, $J^\alpha=(\rho,\vec{J})$, satisfy the field/current identity (FCI)

$$\kappa B \equiv \epsilon^{ij}\partial_i A^j = -e\varrho, \qquad \kappa E^i \equiv -\partial_i A^0 - \partial_t A^i = e\,\epsilon^{ij}J^j. \tag{2}$$

The current and the fields are coupled according to $\varrho=\Psi^*\Psi$ and $\vec{J}=\frac{1}{2im}[\Psi^*\vec{D}\Psi-\Psi(\vec{D}\Psi)^*]$, where $D_\alpha = \partial_\alpha - ieA_\alpha$. For the special value $\Lambda = e^2/m\kappa$ of the non-linearity, the static second-order equation (1) can be reduced to the first-order equations $\big(D_1 \pm iD_2\big)\Psi = 0$. In a suitable gauge, this leads to Liouville's equation and can therefore be solved explicitly [1]. Unlike the vortices in the Abelian Higgs model [2], the Jackiw-Pi vortices have non-zero electric as well as magnetic fields.

In this paper, we present new, *purely magnetic*, non-relativistic solutions, constructed from a spin $\frac{1}{2}$ (rather than scalar) field with Chern-Simons coupling. For this reason, we call them *spinor vortices*. Being purely magnetic, our solutions differ from those found by Leblanc et al. [3], who construct their solutions by putting together the JP solutions and their superpartners using supersymmetry. Our vortices appear to be rather the non-relativistic counterparts of the relativistic fermionic vortices found by Cho et al. [4].

Following Lévy-Leblond [5], we describe non-relativistic, spin $\frac{1}{2}$ fields by the $2+1$ dimensional version of the non-relativistic 'Dirac' equation

$$\begin{cases} (\vec{\sigma}\cdot\vec{D})\,\Phi & + & 2m\,\chi & = & 0,\\ D_t\,\Phi & + & i(\vec{\sigma}\cdot\vec{D})\,\chi & = & 0, \end{cases} \tag{3}$$

where Φ and χ are two-component 'Pauli' spinors and $(\vec{\sigma}\cdot\vec{D}) = \sum_{j=1}^{2}\sigma^j D_j$ with σ^j denoting the Pauli matrices. These spinors are coupled to the Chern-Simons gauge fields through the mass (or particle) density [5]: $\varrho = |\Phi|^2$, as well as through the spatial components of the current, $\vec{J} = i\big(\Phi^\dagger\vec{\sigma}\,\chi - \chi^\dagger\vec{\sigma}\,\Phi\big)$, so that the system (2)-(3) is self consistent. Let us mention that this coupled Lévy-Leblond – Chern-Simons system can be derived from a 3+1 dimensional *massless* Dirac – Chern-Simons system by a light-like dimensional reduction, in a way similar to the one we used for a scalar field [6]. This reduction 'commutes' with the four-dimensional chirality operator

$$\gamma^5 = \begin{pmatrix} -i\sigma_3 & 0\\ 0 & i\sigma_3 \end{pmatrix}, \tag{4}$$

so that Φ and χ are *not* the chiral components of the four-component spinor field $\psi = \begin{pmatrix}\Phi\\ \chi\end{pmatrix}$; these latter are actually defined by $\frac{1}{2}(1 \pm i\gamma^5)\psi_\pm = \pm\psi_\pm$. Independently of this derivation it is easy to see that Eq. (3) splits into two uncoupled systems for the two two-component spinor fields ψ_+ and ψ_-. Each of the chiral components separately describe (in general different) physical phenomena in $2+1$ dimensions. For the ease of presentation, we keep, nevertheless, all four components of ψ.

Now the current can be written using (3) in the form:

$$\vec{J} = \frac{1}{2im}\Big(\Phi^\dagger \vec{D}\Phi - (\vec{D}\Phi)^\dagger\Phi\Big) + \vec{\nabla}\times\Big(\frac{1}{2m}\,\Phi^\dagger\sigma_3\Phi\Big). \tag{5}$$

Using $(\vec{D}\cdot\vec{\sigma})^2 = \vec{D}^2 + eB\sigma_3$, we find that the component-spinors satisfy

$$\left\{\begin{array}{rcl} iD_t\Phi &=& -\frac{1}{2m}\Big[\vec{D}^2 + eB\sigma_3\Big]\Phi, \\ iD_t\chi &=& -\frac{1}{2m}\Big[\vec{D}^2 + eB\sigma_3\Big]\chi - \frac{e}{2m}\,(\vec{\sigma}\cdot\vec{E})\,\Phi. \end{array}\right. \tag{6}$$

Thus, Φ solves a 'Pauli equation', while χ couples through the Darwin term, $\vec{\sigma}\cdot\vec{E}$. Expressing $\vec{E}$ and B through the FCI, (2) and inserting into our equations, we get finally

$$\left\{\begin{array}{rl} iD_t\Phi = & \Big[-\frac{1}{2m}\,\vec{D}^2 + \frac{e^2}{2m\kappa}\,|\Phi|^2\,\sigma_3\Big]\Phi, \\ iD_t\chi = & \Big[-\frac{1}{2m}\,\vec{D}^2 + \frac{e^2}{2m\kappa}\,|\Phi|^2\,\sigma_3\Big]\chi - \frac{e^2}{2m\kappa}\,\big(\vec{\sigma}\times\vec{J}\big)\Phi. \end{array}\right. \tag{7}$$

If the chirality of ψ is restricted to $+1$ (or -1), this system describes non-relativistic spin $+\frac{1}{2}$ $(-\frac{1}{2})$ fields interacting with a Chern-Simons gauge field. Leaving the chirality of ψ unspecified, it describes *two* spinor fields of spin $\pm\frac{1}{2}$, interacting with each other and the Chern-Simons gauge field.

Since the lower component is simply $\chi = -(1/2m)(\vec{\sigma}\cdot\vec{D})\Phi$, it is enough to solve the Φ-equation. For

$$\Phi_+ = \begin{pmatrix}\Psi_+\\ 0\end{pmatrix} \qquad \text{and} \qquad \Phi_- = \begin{pmatrix}0\\ \Psi_-\end{pmatrix} \tag{8}$$

respectively — which amounts to working with the $\pm$ chirality components — the 'Pauli' equation for Φ reduces to

$$iD_t\Psi_\pm = \Big[-\frac{\vec{D}^2}{2m} \pm \lambda\,(\Psi_\pm^\dagger\Psi_\pm)\Big]\Psi_\pm, \qquad \lambda \equiv \frac{e^2}{2m\kappa}, \tag{9}$$

which coincide with Eq. (1), but with non-linearities $\pm\lambda$ — the *half* of the special value used by Jackiw and Pi. For this reason, our solutions (presented below) will be *purely magnetic*, $(A_t \equiv 0)$, unlike in the case studied by Jackiw and Pi.

In detail, let us consider the static system

$$
\left\{
\begin{aligned}
&\Big[-\frac{1}{2m}(\vec{D}^2+eB\sigma_3)-eA_t\Big]\Phi=0,\\
&\vec{J}=-\frac{\kappa}{e}\vec{\nabla}\times A_t,\\
&\kappa B=-e\varrho,
\end{aligned}
\right.
\tag{10}
$$

and try the first-order 'self-dual' Ansatz

$$
\big(D_1\pm iD_2\big)\Phi=0.
\tag{11}
$$

Eq. (11) makes it possible to replace $\vec{D}^2=D_1^2+D_2^2$ by $\mp eB$, then the first equation in (10) can be written as

$$
\Big[-\frac{1}{2m}eB(\mp 1+\sigma_3)-eA_t\Big]\Phi=0,
\tag{12}
$$

while the current is

$$
\vec{J}=\frac{1}{2m}\vec{\nabla}\times\Big[\Phi^\dagger(\mp 1+\sigma_3)\Phi\Big].
\tag{13}
$$

Now, due to the presence of σ_3, Eq. (13) and the second equation in (10) can be solved with a *zero* A_t and $\vec{J}$: by choosing $\Phi\equiv\Phi_+$ ($\Phi\equiv\Phi_-$) for the upper (lower) cases respectively makes $(\mp 1+\sigma_3)\Phi$ vanish. (It is readily seen from Eq. (12) that any solution has a definite chirality).

The remaining task is to solve the SD conditions

$$
(D_1+iD_2)\Psi_+=0,\qquad\text{or}\qquad(D_1-iD_2)\Psi_-=0,
\tag{14}
$$

and $B=-(e\varrho)/\kappa$, where $\varrho=|\Psi_+|^2$ (or $|\Psi_-|^2$ respectively). In the gauge $\Psi_\pm=\varrho^{1/2}$, this yields [1]

$$
\vec{A}=\pm\frac{1}{2e}\vec{\nabla}\times\ln\varrho.
\tag{15}
$$

Thus it reduces, in both cases, to the Liouville equation

$$
\triangle\ln\varrho=\pm\frac{2e^2}{\kappa}\varrho.
\tag{16}
$$

A normalizable solution is obtained for Ψ_+ when $\kappa<0$, and for Ψ_- when $\kappa>0$. (These correspond precisely to having an attractive non-linearity in Eq. (9)). The lower components vanish in both cases, as seen from $\chi=-\frac{1}{2m}(\vec{\sigma}\cdot\vec{D})\Phi$. Both solutions only involve *one* of the $2+1$ dimensional spinor fields $\psi_\pm$, depending on the sign of κ.

It is worth to point out that inserting $A_t = 0$ into the first equation in (10) yields the same equation as the one solved in Ref. [1]. Remember, however, that in the Jackiw - Pi construction, the FCI requires an electric field such that the $eA_t\Psi$ term coming from the covariant derivative $D_t\Psi$ cancels *half* of the non-linearity. Here we obtain the same effective theory with $A_t = 0$ and hence $\vec{E} \equiv 0$. Thus, our solutions are 'purely' magnetic. Furthermore, since the total magnetic flux, $\int d^2x\, B = -(e/\kappa)\int d^2x \varrho \equiv -eN/\kappa$, is nonzero if $N \neq 0$, we call our objects (spinor) vortices.

Eq. (16) can be analyzed following Ref. [1]. The general solution of the Liouville equation is $\varrho = \mp(4\kappa/e^2)|f'(z)|^2(1+|f(z)|^2)^{-2}$ with $f(z)$ complex analytic. A radially symmetric solution is provided, for example, [1] by:

$$\varrho_n(r) = \mp\frac{4n^2\kappa}{e^2r^2}\left((\frac{r_0}{r})^n + (\frac{r}{r_0})^n\right)^{-2}, \qquad z = re^{i\theta}, \tag{17}$$

where r_0 and n are two free parameters. (The single valuedness of $\Psi_\pm$ requires n to be an integer, though [1]). Integrating ϱ_n over all two-space yields $N = 4\pi n|\kappa|/e^2$.

The physical properties as symmetries and conserved quantities can be studied by noting that our equations are in fact obtained by variation of the $2+1$-dimensional action $\int d^3x\mathcal{L}$ with

$$\begin{aligned} &\mathcal{L} = \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + \\ &\Im\left\{\left[\psi_+^\dagger\left(\Sigma_+^t D_t + \Sigma_+^i D_i - 2im\Sigma_+^m\right)\psi_+\right] + \left[\psi_-^\dagger\left(\Sigma_-^t D_t + \Sigma_-^i D_i - 2im\Sigma_-^m\right)\psi_-\right]\right\}\end{aligned} \tag{18}$$

where, with some abuse of notation, we identified the chiral components $\psi_\pm$ with 2-component spinors and introduced the 2×2 matrices

$$\begin{aligned} &\Sigma_+^t = \Sigma_-^t = \tfrac{1}{2}(1+\sigma_3), \qquad &&\Sigma_+^m = \Sigma_-^m = \tfrac{1}{2}(1-\sigma_3), \\ &\Sigma_+^1 = -\sigma_2, \quad \Sigma_+^2 = \sigma_1, &&\Sigma_-^1 = -\sigma_2, \quad \Sigma_-^2 = -\sigma_1. \end{aligned} \tag{19}$$

Note that the matter action decouples into chiral components, consistently with the decoupling of the Lévy-Leblond equation (3).

The action (18) can be used to show that the coupled Lévy-Leblond — Chern-Simons system is, just like its scalar counterpart, Schrödinger symmetric [1], proving that our theory is indeed non-relativistic. A conserved energy-momentum tensor can be constructed and used to derive conserved quantities. One finds that the 'particle number' N determines the actual values of all the conserved charges: for (17), e.g., the magnetic flux, $-eN/\kappa$, and the mass, $\mathcal{M} = mN$, are the same as for the scalar soliton of [1]. The total angular momentum, however, can be shown to be $I = \mp N/2$, *half* of the corresponding value for the scalar soliton. As a consequence of self-duality, our solutions have *vanishing energy*, just like the ones of Ref. [1].

Our vortices are hence similar to the non-relativistic scalar solitons of Jackiw and Pi. On the other hand, the similar flux/angular momentum ratio and the vanishing A_t indicate that they are just as akin to the relativistic fermionic vortices found by Cho et al. [4].

Physical applications as the fractional quantum Hall effect [7] would require to extend the theory to background fields. Solutions in a background uniform magnetic or harmonic force field can be obtained by 'exporting' the empty-space solution, just like for scalars [8]. Let $\mathcal{A}_t \equiv -U$ and $\vec{\mathcal{A}}$ denote the potentials of an external electromagnetic field and consider the modified Lévy-Leblond equation

$$\left\{\begin{array}{lllllll} (\vec{\sigma}\cdot\vec{D}^{ext})\,\Phi^{ext} & & & + & 2m\,\chi^{ext} & = & 0, \\ D_t^{ext}\,\Phi^{ext} & + & \frac{ie}{4m}\mathcal{B}\sigma_3\Phi^{ext} & + & i(\vec{\sigma}\cdot\vec{D}^{ext})\,\chi^{ext} & = & 0, \end{array}\right. \tag{20}$$

where $\mathcal{B} = \vec{\nabla}\times\vec{\mathcal{A}}$ is the external magnetic field and the new covariant derivative is $D_\alpha^{ext} = \partial_\alpha - ieA_\alpha - ie\mathcal{A}_\alpha$. Note that we have also included the anomalous term $\frac{ie}{4m}\mathcal{B}\sigma_3\Phi^{ext}$, which is the reduction of the anomalous term $\frac{1}{16}\mathcal{F}_{ij}[\gamma^i,\gamma^j]\gamma^t\psi$ arising in $3+1$ dimensions [6]. Then the 'upper' component Φ^{ext} solves the 'Pauli' equation with anomalous magnetic moment

$$iD_t^{ext}\,\Phi^{ext} = -\frac{1}{2m}\Big[\big(\vec{D}^{ext}\big)^2 + eB\sigma_3\Big]\Phi^{ext} - \frac{e\mathcal{B}}{4m}\sigma_3\Phi^{ext}. \tag{21}$$

In a constant $\mathcal{B}$-field, for example, set $\mathcal{A}_i = -\frac{1}{2}\epsilon_{ij}\mathcal{B}x^j$, and

$$\psi^{ext}(\vec{x},t) =$$

$$\frac{e^{-(im\omega r^2\tan\omega t)/2}}{\cos\omega t}\begin{pmatrix} e^{i\omega t\sigma_3/2} & 0 \\ i\frac{\omega}{2}\Big[\tan\omega t(\vec{\sigma}\cdot\vec{x}) - (\vec{\sigma}\times\vec{x})\Big]e^{i\omega t\sigma_3/2} & \frac{e^{i\omega t\sigma_3/2}}{\cos\omega t}\end{pmatrix}\psi^0(\vec{X},T), \tag{22}$$

$$A_\alpha^{ext} = \partial_\alpha X^\beta A_\beta^0,$$

with

$$\vec{X} = [\cos\omega t]^{-1}R^{-1}(\omega t)\,\vec{x}, \qquad T = \omega^{-1}\tan\omega t. \tag{23}$$

Here $\omega = e\mathcal{B}/2m$ and $R(\theta)$ is the matrix of a rotation by angle θ in the plane: $\vec{\sigma}\cdot(R\vec{x}) = a(\vec{\sigma}\cdot\vec{x})a^{-1}$ with $a(\theta) \equiv \exp(i\theta\sigma_3/2)$. Then a straightforward calculation shows that (22) solves the external-field equation (20) whenever (ψ^0, A_α^0) is a solution in 'empty' space i.e. with no external field. Note that while the 'upper' component Φ transforms with the same 'time-dependent dilation' factor $[\cos\omega t]^{-1}$ as a scalar field, the 'lower' component, χ, has a conformal factor $[\cos\omega t]^{-2}$ as well as an inhomogenous part, which is linear in Φ. The time derivative of the rotation matrix in Eq. (22) compensates in particular the anomalous term in (21), while in Eq. (20) for this compensation the inhomogenous part of the lower component is also needed. Applying the transformation (22) to the static spinor vortices Eqs. (17,15) yields in particular time-dependent background-field solutions with non-vanishing 'lower' component as well as a (Chern Simons) electric field.

Let us mention in conclusion, that the derivation of the coupled Lévy-Leblond – Chern-Simons system based on light-like dimensional reduction makes also transparent its Schrödinger symmetry. In this procedure, the four-dimensional chirality operator, Eq. (4), and the anomalous term in the external-field, (20), arise naturally. Finally, the external-field equation (20) and the 'solution-exporting' formula (22) both have a nice geometric meaning. These points are explained in Ref. [6].

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